Topic: Inverse trig relations

Question: Which of the following is the entire set of angles α for which $\cos\alpha = -1/2$?

Answer choices:

A The set of all angles α for which there is some integer n such that

$$\alpha = \frac{\pi}{3} + 2n\pi \quad \text{or} \quad \alpha = -\frac{2\pi}{3} + 2n\pi$$

B The set of all angles α for which there is some integer n such that

$$\alpha = \frac{4\pi}{3} + 2n\pi \quad \text{or} \quad \alpha = -\frac{\pi}{3} + 2n\pi$$

C The set of all angles α for which there is some integer n such that

$$\alpha = \frac{2\pi}{3} + 2n\pi \quad \text{or} \quad \alpha = -\frac{2\pi}{3} + 2n\pi$$

D The set of all angles α for which there is some integer n such that

$$\alpha = -\frac{4\pi}{3} + 2n\pi \quad \text{or} \quad \alpha = \frac{2\pi}{3} + 2n\pi$$

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Solution: C

Recall that for any y in the interval $[-1,1]$ and any angle α, $\cos\alpha = y$ if and only if there exists an angle θ with $\cos\theta = y$ and an integer n such that

$$\alpha = \theta + 2n\pi \qquad \text{or} \qquad \alpha = -\theta + 2n\pi$$

Now recall that $\cos(\pi/3) = 1/2$, and note that

$$\frac{2\pi}{3} = \pi - \frac{\pi}{3}$$

so

$$\cos\left(\frac{2\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right)$$

By the difference formula for cosine,

$$\cos\left(\frac{2\pi}{3}\right)$$

$$\cos\left(\pi - \frac{\pi}{3}\right)$$

$$\cos\pi\cos\left(\frac{\pi}{3}\right) + \sin\pi\sin\left(\frac{\pi}{3}\right)$$

$$(-1)\left(\frac{1}{2}\right) + (0)\sin\left(\frac{\pi}{3}\right)$$

$$-\frac{1}{2}$$

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Setting θ to $2\pi/3$ and y to $-1/2$, we find that $\cos\alpha = -1/2$ if and only if there is some integer n such that

$$\alpha = \frac{2\pi}{3} + 2n\pi \qquad \text{or} \qquad \alpha = -\frac{2\pi}{3} + 2n\pi$$

If we let $n = 0$ in the first clause above, we get $\alpha = 2\pi/3$. And if we let $n = 1$ in the second clause, we get

$$\alpha = -\frac{2\pi}{3} + 2(1)\pi = -\frac{2\pi}{3} + 2\pi = \frac{1(-2\pi) + 3(2\pi)}{3} = \frac{-2\pi + 6\pi}{3} = \frac{4\pi}{3}$$

We have found that both $\alpha = 2\pi/3$ and $\alpha = 4\pi/3$ must be included in the answer. Note that both $\alpha = 2\pi/3$ and $\alpha = 4\pi/3$ are in the interval $[0,2\pi)$. Since there can be only two angles in that interval whose cosine is equal to $-1/2$, we see that $2\pi/3$ and $4\pi/3$ are the *only* angles in the interval $[0,2\pi)$ whose cosine is equal to $-1/2$.

Now we'll show that none of the other three answer choices is correct.

If we let $n = 0$ in the first clause in answer choice A, we get $\alpha = \pi/3$. Note that $\pi/3$ is in the interval $[0,2\pi)$, and that $\pi/3$ is equal to neither $2\pi/3$ nor $4\pi/3$. Thus the set of angles given in answer choice A includes angles α for which $\cos\alpha \neq -1/2$.

If we let $n = 1$ in the second clause in answer choice B, we get

$$\alpha = -\frac{\pi}{3} + 2(1)\pi = -\frac{\pi}{3} + 2\pi = \frac{1(-\pi) + 3(2\pi)}{3} = \frac{-\pi + 6\pi}{3} = \frac{5\pi}{3}$$

Note that $5\pi/3$ is in the interval $[0,2\pi)$, and that $5\pi/3$ is equal to neither $2\pi/3$ nor $4\pi/3$. Thus the set of angles given in answer choice B includes angles α

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Question: Which of the following is the entire set of angles α for which $\sin\alpha = \sqrt{2}/2$?

Answer choices:

A The set of all angles α for which there is some integer n such that

$$\alpha = 45^\circ + n(360^\circ) \quad \text{or} \quad \alpha = -315^\circ + n(360^\circ)$$

B The set of all angles α for which there is some integer n such that

$$\alpha = 45^\circ + n(360^\circ) \quad \text{or} \quad \alpha = 135^\circ + n(360^\circ)$$

C The set of all angles α for which there is some integer n such that

$$\alpha = 225^\circ + n(360^\circ) \quad \text{or} \quad \alpha = 45^\circ + n(360^\circ)$$

D The set of all angles α for which there is some integer n such that

$$\alpha = -45^\circ + n(360^\circ)\pi \quad \text{or} \quad \alpha = -135^\circ + n(360^\circ)$$

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Solution: B

Recall that for any y in the interval $[-1,1]$ and any angle α, $\sin\alpha = y$ if and only if there exists an angle θ with $\sin\theta = y$ and an integer n such that

$$\alpha = \theta + n(360^\circ) \qquad \text{or} \qquad \alpha = 180^\circ - \theta + n(360^\circ)$$

Recall that $\sin(45^\circ) = \sqrt{2}/2$. Setting θ to 45° and y to $\sqrt{2}/2$, we find that $\sin\alpha = \sqrt{2}/2$ if and only if there is some integer n such that

$$\alpha = 45^\circ + n(360^\circ) \qquad \text{or} \qquad \alpha = 180^\circ - 45^\circ + n(360^\circ)$$

Since $180^\circ - 45^\circ = 135^\circ$, this can be written as

$$\alpha = 45^\circ + n(360^\circ) \qquad \text{or} \qquad \alpha = 135^\circ + n(360^\circ)$$

If we let $n = 0$ in the first clause above, we get $\alpha = 45^\circ$. And if we let $n = 0$ in the second clause, we get 135°.

We have found that both $\alpha = 45^\circ$ and $\alpha = 135^\circ$ must be included in the answer. Note that both 45° and 135° are in the interval $[0^\circ, 360^\circ)$. Since there can be only two angles in that interval whose sine is equal to $\sqrt{2}/2$, we see that 45° and 135° are the *only* angles α in the interval $[0^\circ, 360^\circ)$ whose sine is equal to $\sqrt{2}/2$.

Now we'll show that none of the other three answer choices is correct.

If we let $n = 0$ in the first clause in answer choice A, we get $\alpha = 45^\circ$. And if we let $n = 1$ in the second clause in answer choice A, we get

$$\alpha = -315^\circ + 1(360^\circ) = -315^\circ + 360^\circ = 45^\circ$$

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Note that every angle α that satisfies the first clause in answer choice A must be coterminal with $45°$, and that the same is true for every angle α that satisfies the second clause in answer choice A. Since $135°$ isn't coterminal with $45°$, the set of angles given in answer choice A is incomplete, because it doesn't include $\alpha = 135°$.

If we set n to 0 in the first clause in answer choice C, we get $\alpha = 225°$. Note that $225°$ is in the interval $[0°,360°)$, and that $225°$ is equal to neither $45°$ nor $135°$. Thus the set of angles given in answer choice C includes angles α for which $\sin\alpha \neq \sqrt{2}/2$.

If we set n to 1 in the first clause in answer choice D, we get

$$\alpha = -45° + 1(360°) = -45° + 360° = 315°$$

Note that $315°$ is in the interval $[0,360°)$, and that $315°$ is equal to neither $45°$ nor $135°$. Thus the set of angles given in answer choice D includes angles α for which $\sin\alpha \neq \sqrt{2}/2$.

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Topic: Inverse trig relations

Question: Express m as a function of n, assuming that the sine of an angle α (in radians) is equal to -1 if and only if there are integers n, m such that

$$\frac{3\pi}{2} + 2n\pi = \alpha = -\frac{9\pi}{2} + 2m\pi$$

Answer choices:

A $m = n + 1$

B $m = 2 - 3n$

C $m = 3 + n$

D $m = n - 2$

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Solution: C

What we need to do is take the equation

$$\frac{3\pi}{2} + 2n\pi = -\frac{9\pi}{2} + 2m\pi$$

and solve it for m.

Adding $9\pi/2$ to both sides, we have

$$\frac{9\pi}{2} + \frac{3\pi}{2} + 2n\pi = 2m\pi$$

Now

$$\frac{9\pi}{2} + \frac{3\pi}{2} = \frac{12\pi}{2} = 6\pi$$

so

$$6\pi + 2n\pi = 2m\pi$$

Dividing both sides by 2π, we get

$$3 + n = m$$

To be on the safe side, we'll show that our assumption is true.

First, let's show that $\sin\alpha = -1$ if there is some integer n such that

$$\alpha = \frac{3\pi}{2} + 2n\pi$$

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Well, we know that $\sin(3\pi/2) = -1$, so $\sin\alpha = -1$ if α is an angle that's coterminal with $3\pi/2$, hence $\sin\alpha = -1$ if there is some integer n such that

$$\alpha = \frac{3\pi}{2} + 2n\pi$$

Next, let's show that $\sin\alpha = -1$ if there is some integer m such that

$$\alpha = -\frac{9\pi}{2} + 2m\pi$$

To do this, we'll first show that

$$\sin\left(\frac{11\pi}{2}\right) = -1$$

Well,

$$\frac{11\pi}{2} = \frac{8\pi + 3\pi}{2} = 4\pi + \frac{3\pi}{2}$$

which differs from an angle of $3\pi/2$ by 4π (a multiple of 2π), so

$$\sin\left(\frac{11\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) = -1$$

Also, we know that (in general) if y is an element of $[-1,1]$, then $\sin\alpha = -1$ if there exist some angle θ such that $\sin\theta = -1$ and some integer m such that

$$\alpha = \pi - \theta + 2m\pi$$

Setting θ to $11\pi/2$, we find that $\sin\alpha = -1$ if there exists some integer m such that

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$$\alpha = \pi - \frac{11\pi}{2} + 2m\pi$$

Now

$$\pi - \frac{11\pi}{2} = \frac{2(\pi) - 1(11\pi)}{2} = \frac{2\pi - 11\pi}{2} = -\frac{9\pi}{2}$$

Therefore, $\sin\alpha = -1$ if there is some integer m such that

$$\alpha = -\frac{9\pi}{2} + 2m\pi$$

Now recall that there is only one angle α in the interval $[0,2\pi)$ (hence there is only one angle α in any interval of length 2π) such that $\sin\alpha = -1$, so $\sin\alpha = -1$ if and only if there is an integer n such that

$$\alpha = \frac{3\pi}{2} + 2n\pi$$

and $\sin\alpha = -1$ if and only if there is an integer m such that

$$\alpha = -\frac{9\pi}{2} + 2m\pi$$

Combining these results, we find that $\sin\alpha = -1$ if and only if there are integers n, m such that

$$\frac{3\pi}{2} + 2n\pi = \alpha = -\frac{9\pi}{2} + 2m\pi$$

This shows that the assumption given in the statement of this problem is true.

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